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Why such a meeting ?

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1 Introduction

Since some time I have been interested in such topics as how to engineer *genuine* mathematical concepts on the computer. However, even a proper formulation of the question itself is not yet clear, for there must be a kind of spirals of what we should do, what we have done and what we can do.

The idea comes from very impressive recent developments of mathematical softwares. I do not deny its naiveness. We have numerical computations, symbolic comuptations, and lots of things are now carried out on computer. In fact, we know most modern sophisticated activities are de facto carried out computationally, based upon mathematical or pseudo-mathematical principles. Nevertheless, you must have observed plenty of discrepancies in the mathematics-computer relations. We thus have to build a synthesized world of mathematical reasoning and computations, hoping that its impact does not remain within a mathematico-technical sphere.

As the first step, I have been trying to show relevance of developing a computer version of classical mathematical analysis, in which crucial roles are played by various concepts of infinities and topologies, crystalized in the form of inequalities (See, e.g., [10, 11]). However, such an endeavor can only become possible with cooperations of brave people, amateurs of adventures. Please pay your particular attention to the contributions of these courageous mathematicians in these Proceedings.

Finally, to add some academic flavor to my notes, I show a further example of transferrability of good mathematical reasoning into a computer-related world (For another example, see [11] with respect to Calderón's work [3]).

In the classical measure theory, a regularity property of integration is expressed in terms of the so called differentiation property or density theorem. The standard technique employed therein is a class of covering theorems (See

Guzmán[4]). Covering theorems of Vitali [8] and Besicovitch [1, 2] are typical instances. While Vitali's covering theorem is proven by a transfinite means such as Hausdorff's maximum principle, Besicovitch's theorem admits a very algorithmic demonstration. This fact is, in the context of *philosophy*, say, very interesting, for Vitali's theorem is valid for the Lebesgue measure and Bescovitch's theorem for the Radon measure.

Here we show in fact the customary proof of Bescovitch's theorem admits an algorithmic interpretation. We in fact implement such algorithms into Maple V procedures, and are ready for experimenting in the case of finite coverings, though possibly and hopefully with high enough cardinalities. In §5, we only compute a family of 1000 disks. The efficiency also depends on the preassigned ranges of values of radii and how they are scattered.

Note that the idea of covering theorems contains a reduction to a simpler structure from a very complicated one [1], of any cardinality. A most general discussion involved therein requires the Axiom of Choice. It should thus be worth emphasizing that actual computer experiments can be carried out retaining essential algorithmic parts of the proof.

2 The Besicovitch Covering theorem

Let $\mathcal{A}$ be a family of closed disks in the Euclid plane $\mathbb{R}^2$. For a disk $D \in \mathcal{A}$ we denote by center(D) and radius(D) respectively its center and radius. Thus, when $c = \text{center}(D)$ and $r = \text{radius}(D)$,

$$D = D_r(c) = \{ \ x \in \mathbb{R}^2 \ ; \ |x - c| \leq r \ \}.$$

Here $|\ |$ stands for the Euclid distance. When $\mathcal{A}$ is a countable family, the disks therein can be numbered:

$$\mathcal{A} = \{ \ D_n \ ; \ n = 1, 2, \cdots \},$$

whence, for each $n = 1, 2, \cdots$, we may speak of

$$\mu_n(\mathcal{A}) = \text{the cardinality of } \{ \ D_j \ ; D_j \cap D_n \neq \emptyset, \ j < n\}. \tag{1}$$

Thus, $\mu_n(\mathcal{A})$'s depend on the numbering of $\mathcal{A}$. However,

$$\gamma(\mathcal{A}) = \sup_{n \geq 1} \ \mu_n(\mathcal{A}) \quad (\leq \infty) \tag{2}$$

[1] This is why I am tempted to cite Watts' book[9].

is independent of the numbering of $\mathcal{A}$.

If $\mathcal{A}$ is not countable, then this definiton of $\gamma(\mathcal{A})$ is no more valid. However, the Besicovitch covering theorem ensures a certain uniform bound β such that any family $\mathcal{B}$ of closed disks has a countable subfamily $\mathcal{D}$ with $\gamma(\mathcal{D}) \leq \beta$. More precisely, we have the following

Theorem. *Let $\mathcal{B}$ be a family of closed disks (of any cardinality > 0). Suppose*

$$R = \sup \{ \text{ radius}(B) \ ; \ B \in \mathcal{B} \} < +\infty. \tag{3}$$

Let

$$\mathcal{C} = \{ \text{ center}(B) \ ; \ B \in \mathcal{B} \} \tag{4}$$

be the set of the centers of the disks in $\mathcal{B}$. There exists a countable subfamily $\mathcal{D}$ of $\mathcal{B}$ such that $\gamma(\mathcal{D}) \leq \beta$ and

$$\mathcal{C} \subset \bigcup_{B \in \mathcal{D}} B. \tag{5}$$

Here $\beta \in \mathbb{N}$ and is independent of the choice of $\mathcal{B}$ and of R. Furthermore, there are at most $\beta + 1$ subfamilies $\mathcal{D}_n$, $n = 1, \cdots, \beta + 1$, such that

$$\mathcal{D} = \bigcup_{n=1}^{\beta+1} \mathcal{D}_n. \tag{6}$$

Moreover, each $\mathcal{D}_n$ consists of disjoint closed disks.

For a proof, see, e.g., Besicovitch [1, 2], Ziemer[12] or Evans et al.[5]. See also Guzmán[4].

In the present note, we are mainly interested in an algorithm for extracting a subfamily $\mathcal{D}$ from $\mathcal{B}$ and another for selecting $\mathcal{D}_n$'s from $\mathcal{D}$. Actually, the proofs cited above are basically carried out along similar algorithms. In implementing such algorithms, we assume $\mathcal{B}$ is *finite*, though its cardinality $\#\mathcal{B}$ can be sufficiently high compared to the size of β.

Remark. A rough estimate $\beta \leq 289$ is known. A more detailed discussion on its estimate will be given elsewhere. We also note that analogous results to Theorem hold in the case of $\mathbb{R}^n$ and/or for other regular geometric objects than closed disks.

3 Ideas of algorithms

Suppose we are given a *finite* family $\mathcal{E}$ of closed disks :

$$\mathcal{E} = \{E_n \ ; \ E_n = D_{\rho_n}((\xi_n, \eta_n)), \ n = 1, \cdots, N \}. \tag{7}$$

Here, the cardinality of $\mathcal{E}$ is thus $N = \#\mathcal{E}$. We have to choose a closed disk F from $\mathcal{E}$ and then assign a subfamily $\mathcal{G}$ of $\mathcal{E}$. Thus,

$$\mathbf{P}_0 \ : \ \mathcal{E} \to F \tag{8}$$
$$\mathbf{P}_1 \ : \ \mathcal{E} \to \mathcal{G} \tag{9}$$

and their recipes will be specified below.

Here are some basic ingredients. Let n be a positive integer $\leq \#\mathcal{E}$ and $D = D_r((u,v))$ a closed disk. We will require the set

$$S^*(D,n,\mathcal{E}) = \{ \ 1 \leq j < n \ ; \ (\xi_j - u)^2 + (\eta_j - v)^2 > (\rho_j + r)^2 \ \}$$

of the indices j ($< n$) of disks in $\mathcal{E}$ not intersecting with D and the set

$$C^*(D,\mathcal{E}) = \{ \ j \ ; \ (\xi_j - u)^2 + (\eta_j - v)^2 > r^2 \ \}$$

of the indices j of centers of disks in $\mathcal{E}$ outside the disk D. We will also need a certain complementary set of $S^*(D,n,\mathcal{E})$, i.e.,

$$S_*(D,n,\mathcal{E}) = \{ \ 1 \leq j < n \ ; \ (\xi_j - u)^2 + (\eta_j - v)^2 \leq (\rho_j + r)^2 \ \},$$

the set of the indices $< n$ of disks in $\mathcal{E}$ intersecting with D. For $n = 1$, both $S^*(D,n,\mathcal{E})$ and $S_*(D,n,\mathcal{E})$ are empty. Since verifications of equalities are involved, $S_*(D,n,\mathcal{E})$ is less convenient than $S^*(D,n,\mathcal{E})$ from the view point of *computer analysis* (See Pour-El et al.[7]).

The procedure $\mathbf{P}_0$ goes as follows:

P01 Find

$$R = \max \{ \ \rho_i = \text{radius}(E_i) \ ; \ E_i = D_{\rho_i}((\xi_i, \eta_i)) \in \mathcal{E} \ \}.$$

P02 Let m be the smallest index such that $\rho_m > \frac{3}{4}R$.

P03 Let $F = D_{\rho_m}((\xi_m, \eta_m))$.

Remark. Note that we may adopt another selection principle in Step P02.

The procedure $\mathbf{P}_1$ then goes as follows:

P11 Let $F = \mathbf{P}_0(\mathcal{E})$.

P12 Let $\mathcal{E}'$ be the set of the disks in $\mathcal{E}$ with indices belonging to the set $C^*(F, \mathcal{E})$.

P13 Renumber $\mathcal{E}'$ for the sake of normalizing purpose, and let the resulting family be $\mathcal{G}$.

Note $\#\mathcal{G} < \#\mathcal{E}$. If the set $C^*(F, \mathcal{E})$ is empty in Step P12, then $\mathcal{G}$ is empty. Thus, the procedures $\mathbf{P}_1$ cannot be repeated indefinitely.

Therefore, combining the procedures $\mathbf{P}_0$ and $\mathbf{P}_1$, we can extract a sequence of closed disks from a given family $\mathcal{E}$ of closed disks. Namely, we have

$$F_m = \mathbf{P}_0(\mathbf{P}_1)^{m-1}(\mathcal{E}), \quad m = 1, \cdots, M, \tag{10}$$

where $(\mathbf{P}_1)^M(\mathcal{E}) = \emptyset$. This means that we have a procedure $\mathbf{B}$ which produces from a given family $\mathcal{E}$ of closed disks a subfamily $\mathcal{F}$ of closed disks F_m as given by (10):

$$\mathbf{B} \;:\; \mathcal{E} \;\rightarrow\; \mathcal{F}. \tag{11}$$

We call $\mathbf{B}$ a Besicovitch procedure, for $\mathcal{F}$ is an extracted subfamily of $\mathcal{E}$ satisfying the requirement of the Besicovitch Covering Theorem. In fact, the procedures $\mathbf{P}_0$, $\mathbf{P}_1$ and their repeated applications are key steps in the classical proof of this theorem. Observe that $\mathbf{P}_0$ makes sense for $\mathcal{E}$ of any cardinality by the Axiom of Choice, that the set $C^*(F, \mathcal{E})$ is defined conceptually, and that, allowing $M = \infty$, $\mathbf{B}$ can be defined for any family $\mathcal{E}$ of closed disks (with uniformly bounded radii).

Here is another procedure $\mathbf{Q}_0$ producing a nonnegative integer q from a given family $\mathcal{E}$ of closed disks and a positive integer $n < \#\mathcal{E}$:

$$\mathbf{Q}_0 \;:\; (\mathcal{E}, n) \;\rightarrow\; q. \tag{12}$$

$\mathbf{Q}_0$ works in the following way.

Q01 Let $D = E_n \in \mathcal{E}$.

Q02 Let $S_* = S_*(D, n, \mathcal{E})$ and q be its cardinality: $q = \#S_*$.

Remark. If $n = 1$, then S_* in Step Q02 is empty, and $\mathbf{Q}_0$ automatically renders $q = 0$. Note $\mathbf{Q}_0(\mathcal{E}, n) = \mu_n(\mathcal{E})$ (See (1)).

$\mathbf{Q}_0$ naturally induces the procedure $\mathbf{Q}$ as follows :

$$\mathbf{Q}(\mathcal{E}) = \max \{ \ \mathbf{Q}_0(\mathcal{E}, n), \ n = 1, \cdots, \#\mathcal{E} \ \}. \tag{13}$$

Note $\mathbf{Q}$ is defined for any finite family $\mathcal{E}$ of closed disks and $\mathbf{Q}(\mathcal{E}) = \gamma(\mathcal{E})$ (See (2)). However, for a Besicovitch subfamily $\mathcal{F} = \mathbf{B}(\mathcal{E})$, it is known $\mathbf{Q}(\mathcal{F}) \leq \beta$ for some $\beta \leq 289$ regardless of the cardinality of $\mathcal{F}$.

Now we define a procedure $\mathbf{R}$, which returns a positive integer m for a family $\mathcal{E}$ of closed disks and a positive integer $n \leq \#\mathcal{E}$:

$$\mathbf{R} \ : \ (\mathcal{E}, n) \ \rightarrow \ m. \tag{14}$$

$\mathbf{R}$ works in the following way:

R1 Let $\ell = \mathbf{Q}(\mathcal{E}) + 1$.

R2 Let $m = n$ for $n = 1, \cdots, \ell$.

R3 Let $\ell + 1 \leq n \leq \#\mathcal{E}$.

- R31 Let $\mathbf{R}(\mathcal{E}, k)$ be defined for $k < n$.
- R32 Let $D = E_n \in \mathcal{E}$ and $S_* = S_*(D, n, \mathcal{E})$.
- R33 Let $T_* = \{1, \cdots, \ell\} \setminus \{ \ \mathbf{R}(\mathcal{E}, k) \ ; \ k \in S_* \ \}$.
- R34 Let $m = \min \ T_*$.

Remark. Note Step R3 is actually an iteration procedure. Note also that the choice of ℓ in Step R1 is to ensure the iteration procedure in Step R3, and in this respect we can replace the definition of ℓ by any integer larger than $\mathbf{Q}(\mathcal{E}) + 1$. Thus, for the Besicovitch subfamily $\mathcal{F} = \mathbf{B}(\mathcal{E})$, we can modify the procedure $\mathbf{R}$ by choosing $\ell = \beta + 1$ or, e.g., $\ell = 290$, in Step R1 to get $\mathbf{R}(\mathcal{F}, n)$, thus bypassing entirely the procedure $\mathbf{Q}$.

The procedure $\mathbf{R}$ defines a mapping

$$\delta \ : \ \mathcal{E} \ni E_n \ \mapsto \ \mathbf{R}(\mathcal{E}, n) \in \{ \ 1, \cdots, \ell \ \}. \tag{15}$$

δ gives a classification of $\mathcal{E}$ by its inverse images

$$\delta^{-1}(n) = \mathcal{E}_n, \quad n = 1, 2, \cdots, \ell. \tag{16}$$

Moreover, the construction, particularly Step R33, ensures that

$$E' \cap E'' = \emptyset$$

for distinct elements $E',\ E'' \in \mathcal{E}_n,\ n = 1, \cdots, \ell$. We have thus arrived at a certain procedure **BD**:

$$\mathbf{BD}\ :\ (\mathcal{E}, n)\ \to\ \mathcal{E}_n,\ n = 1,\ \cdots,\ \ell. \tag{17}$$

where each $\mathcal{E}_n$ is given by (16).

Now the procedures **B** and **BD** realize a Besicovitch covering for a given *finite* family $\mathcal{B}$ of closed disks. In fact,

$$\mathcal{D} = \mathbf{B}(\mathcal{B}),$$
$$\{\ \mathcal{D}_1,\ \cdots,\ \mathcal{D}_\ell\ \} = \{\ \mathbf{BD}(\mathcal{D}, n)\ , n = 1,\ \cdots,\ \ell\ \}$$

and the set of centers $\mathcal{C} = \{\text{center}(B)\ ;\ B \in \mathcal{B}\ \}$ is covered by disks in $\mathcal{D}_n$:

$$\mathcal{C} \subset \bigcup_{D \in \mathcal{D}} D, \quad \mathcal{D} = \bigcup_{n=1}^{\ell} \mathcal{D}_n.$$

4 Proofs of algorithms and related discussions

That the validity of the procedure **B** is clear. For a family $\mathcal{E}$ of closed disks, an execution of $\mathbf{P}_0(\mathcal{E})$ or $\mathbf{P}_1(\mathcal{E})$ require operations of order $(\#\mathcal{E})^2$. The resulting subfamily $\mathcal{F} = \mathbf{B}(\mathcal{E})$ certainly contains the set of the centers of the disks in $\mathcal{E}$ because of Step P12 in the procedure $\mathbf{P}_1$, which is crucial in designing **B**. See [12] or [5]. When the cardinality of $\mathcal{E}$ is finite, the combined procedure **B** to get $\mathcal{F} = \mathbf{B}(\mathcal{E})$ finishes in finite steps. However, operations required then are of order $(\#\mathcal{E})^3$.

The procedures $\mathbf{Q}_0$ and **Q** concern with the numbers of the closed disks which intersect with the given disk. The procedures rely on an enumeration of elements of $\mathcal{E}$. The procedure **Q** is simple, but its execution requires operations of order $(\#\mathcal{E})^2$.

The procedure **R** is quite complicated. The choice of ℓ is crucial for assuring the feasibility of Step R3. In fact, each $\mathbf{R}(\mathcal{E}, n)$ belongs to $\{\ 1, \cdots, \ell\ \}$ and the cardinality of $S_*(E_n, n, \mathcal{E})$ does not exceed $\ell - 1$. That is, Steps R33 and R34 are executable.

The Besicovitch covering theorem is valid for any family of closed disks $\mathcal{B}$ (satisfying (3)). If the set $\mathcal{C}$ of the centers (4) is bounded, it is known that $\lim_{n\to\infty} \text{radius}(D_n) = 0$ for D_n in a Besicovitch subfamily of $\mathcal{B}$. This means that we will not detect disks with radii below the limit of the resolution. Thus, it is suggested that a certain finiteness is in fact natural. The present trial will then make quite a sense provided those undetected disks can be evaluated in advance by any mathematical means.

5 Implementations of Besicovitch procedures in Maple V

We represent the family $\mathcal{E}$ of closed disks given by (7) as a list E of lists $E[i]$:

$$E = [E[1], E[2], \cdots, E[N]],\ N = \#\mathcal{E},$$

where:

$$E[i] = [[x_i, y_i], r_i],\ [x_i, y_i] = E[i][1],\ r_i = E[i][2].$$

$E[i]$ thus stands for a closed disk with the center (x_i, y_i) and the radius r_i. In order to generate examples of families of *closed disks*, we appeal to random numbers generated by the following procedure (see Managan et al. [6] §2.2) :

```
> uniform:=proc(r::constant..constant)
> local intrange, f:
> intrange:=map(x->round(x*10^Digits), evalf(r)):
> f:=rand(intrange):
> (evalf@eval(f))/10^Digits:
> end:
```

Let

```
> x:=uniform(-10.0..10.0):y:=uniform(-10.0..10.0):
> r:=uniform(0.1..2):
```

Then centers (x, y) and radii r chosen randomly in $-10 \leq x,\ y \leq 10$ and $0.1 \leq r \leq 2$. For instance, a family of 1000 closed disks is now generated in the following way:

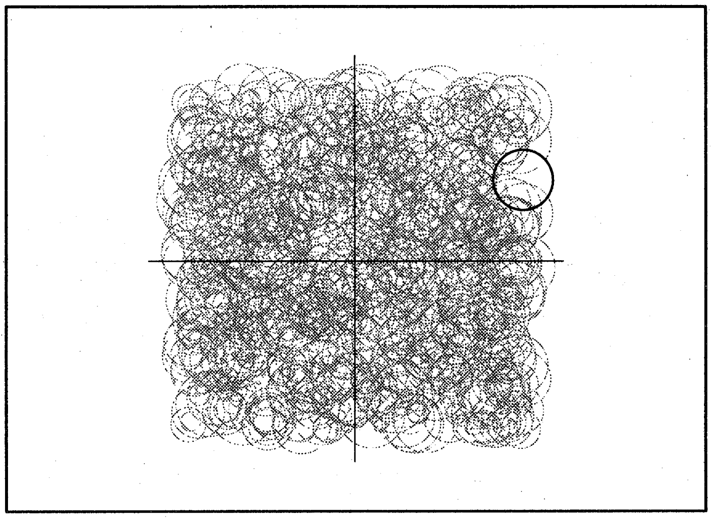
```
> E:=[seq([[x(),y()],r()],i=1..1000)]:
```

Now the following is a Maple V implementation of the procedure $\mathbf{P}_0$:

```
> P0:=proc(E::list)
> local i,R,C1,C2:
> R:=max(seq(E[i][2],i=1..nops(E))):
> C1:={}:
> for i from 1 to nops(E) do
> if E[i][2]>3/4*R then C1:=C1 union {E[i]} fi:
> od:
> C2:=convert(C1,list):
> C2[1]:
```

```
> end:
```

Here is a juxtaposed plot of the family $\mathcal{E}$ of closed disks given in the above and the result of P0(E) as a bold circle.



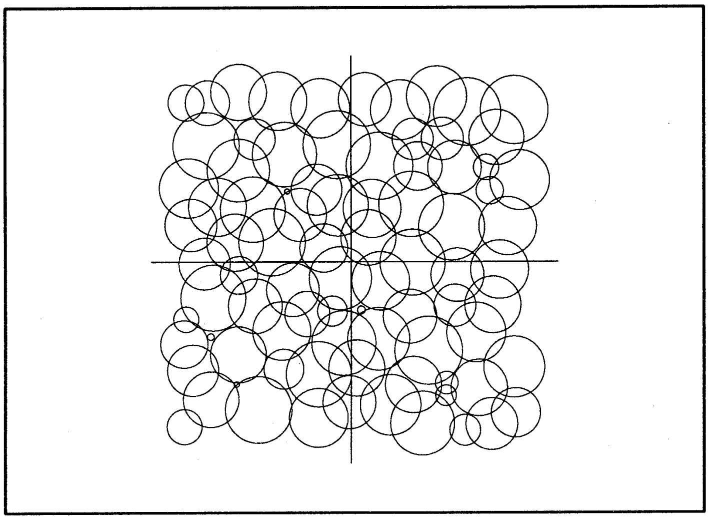
Here is a Maple V implementation of the procedure $\mathbf{P}_1$.

```
> P1:=proc(E::list)
> local i, C,F,r:
> F:=P0(E)[1]:
> r:=P0(E)[2]:
> C:={}:
> for i from 1 to nops(E)
> do
> if (E[i][1][1]-F[1])^2+(E[i][1][2]-F[2])^2>r^2
> then C:=C union {E[i]}
> fi:
> od:
> convert(C,list):
> end:
```

The following is the procedure $\mathbf{B}$ in the Maple V way:

```
> B:=proc(E::list)
> local G,L:
```

```
> G:={}:
> L:=E:
> while nops(L)>0 do
> G:=G union {P0(L)}:
> L:=P1(L):
> od:
> convert(G,list):
> end:
```



Here is a Maple V procedure corresponding to a construction of the set $S_*(D,n,\mathcal{E})$.

```
> S:=proc(D::list, n::posint,E::list)
> local j, s:
> s:={}:
> for j from 1 to n-1 do
> if (E[j][1][1]-D[1][1])^2
> +(E[j][1][2]-D[1][2])^2
> <=(E[j][2]+D[2])^2
> then s:=s union {j}
> fi:
> od:
> s:
```

```
> end:
```

Now the following is a Maple V version of the procedure $\mathbf{Q}_0$ (see (12)).

```
> Q0:=proc(E::list, n::posint)
> local D,s:
> D:=E[n]:
> s:=S(D,n,E):
> nops(s):
> end:
```

Thus, the procedure $\mathbf{Q}$ (see (13)) is implemented as follows:

```
> Q:=proc(E::list)
> local i, N:
> N:=nops(E):
> max(seq(Q0(E,i),i=1..N)):
> end:
```

Now we are ready for giveing a Maple V implementation of the procedure **R** (see (14)).

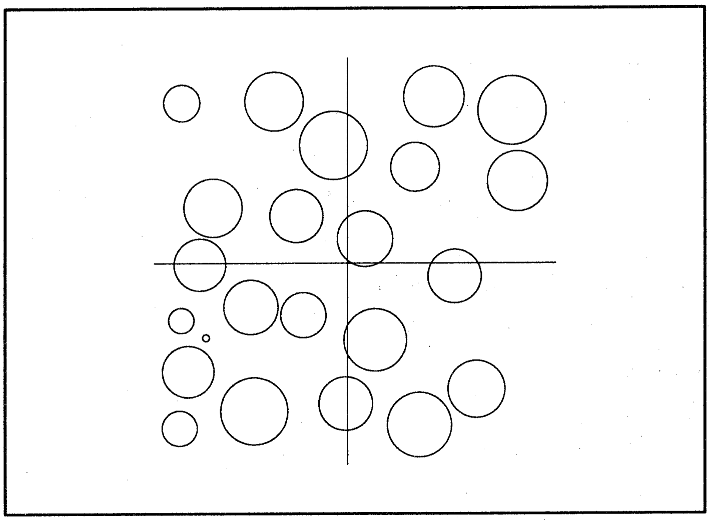
```
> R:=proc(E::list,n::posint)
> local i,j,k,M,MM,N,T,K,L,m:
> K:=nops(E):
> L:=Q(E)+1:
> N:={seq(i,i=1..L)}:
> for i from 1 to L do m[i]:=i od:
> for k from L+1 to K do
> M:=S(E[k],k,E):
> MM:={seq(m[M[i]],i=1..nops(M))}:
> T:=N minus MM:
> m[k]:=min(seq(T[j],j=1..nops(T))):
> od:
> m[n]:
> end:
```

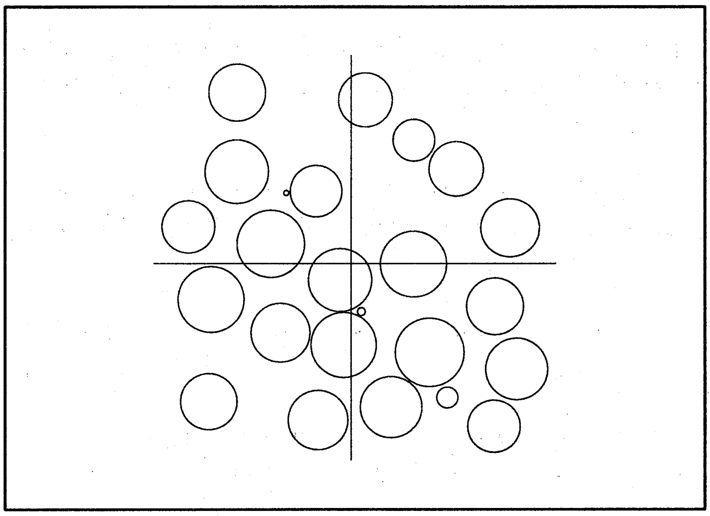
Finally, we have the following Maple V version of the procedures $\mathbf{BD}(E,n)$ (See (17)).

```
> BD:=proc(E::list,n::posint)
> local N,i,j,s:
> if n>Q(E)+1 then [] fi :
> N:=nops(E):
```

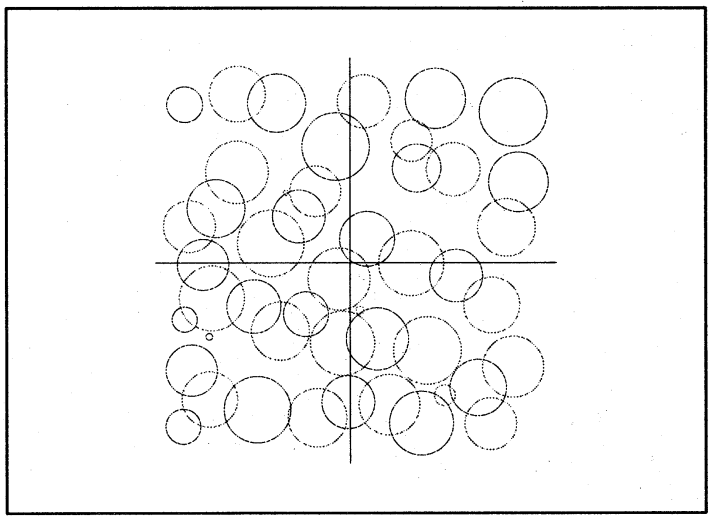
```
> s:={}:
> for i from 1 to N do
> if R(E,i)=n then
> s:=s union {E[i]} fi:
> od:
> convert(s,list):
> end:
```

Here are plots of BD(B(E),n) for $n = 1,\ 2$.





Finally, we juxtapose these two outputs:



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